

Quark mass correction to the string potential

G. Lambiase*, V.V. Nesterenko

Bogoliubov Laboratory of Theoretical Physics

Joint Institute for Nuclear Research, Dubna, 141980, Russia

Abstract

A consistent method for calculating the interquark potential generated by the relativistic string with massive ends is proposed. In this approach the interquark potential in the model of the Nambu–Goto string with point-like masses at its ends is calculated. At first the calculation is done in the one-loop approximation and then the variational estimation is performed. The quark mass correction results in decreasing the critical distance (deconfinement radius). When quark mass decreases the critical distance also decreases. For obtaining a finite result under summation over eigenfrequencies of the Nambu–Goto string with massive ends a suitable mode-by-mode subtraction is proposed. This renormalization procedure proves to be completely unique. In the framework of the developed approach the one-loop interquark potential in the model of the relativistic string with rigidity is also calculated.

PACS number(s): 11.17.+y, 12.38.Aw, 12.40.Aa

1 Introduction

Investigation of the quark interaction at large distances is outside the QCD perturbation theory. Usually, in this field the lattice simulations and string models are used.

Calculation of the quark interaction in the framework of string models has rather long history (see, for example, papers [1] – [9] and references therein). In all these investigations without exception, only the static interquark potential has been considered. It implies that the quarks are assumed to be infinitely heavy. Obviously, this potential, by definition, does not depend on the quark mass. Assumption about infinitely heavy quarks is rather crude at least for u and d quarks with masses about 140 MeV that is significantly less than the characteristic hadronic mass scale ~ 1 GeV. It is clear, that in a general case the interquark potential should depend on quark masses. Both the general approach to this problem in the framework of QCD [10] and the numerical calculations of the light and heavy meson spectra in potential

*Permanent address: Dipartimento di Fisica Teorica e S. M. S. A., Università di Salerno, 84081 Baronissi (SA), Italia

models [11]– [13] testify to this. Certainly, in this case one should say not about the static potential generated, for example, by a relativistic string connecting quarks but simply about the interaction potential between quarks having finite mass rather than infinite one.

The aim of the present paper is an attempt to extend the standard approach to calculation of the interquark potential in the framework of the string models [1], [5] – [9] to the case of the finite quark mass. It turns out that this program can be realized. To this end, the boundary conditions in the string model in question should be modified and a new renormalization procedure should be developed.

In a proposed approach, a correction to the string potential due to the finite quark masses is calculated both in the Nambu–Goto string model and in the Polyakov–Kleinert rigid string model.

In the Nambu–Goto string with massive ends the quark potential is calculated first in one-loop approximation of perturbation theory for arbitrary dimension of the space–time D and then via variational estimation in the limit $(D - 2) \rightarrow \infty$. As known, the static quark potential generated by Nambu–Goto string in one-loop approximation is compiled by two terms, linearly rising confinement potential and universal Lüscher $1/R$ -term [14]. When the ends of the Nambu–Goto string are loaded by point-like masses (quarks) then the potential, in addition to terms mentioned above, acquires correction dependent on quark mass m and the distance between quarks, R . In fact, R and m are involved in potential correction in the form of a dimensionless parameter $q = (M_0/m)M_0 R$, where M_0^2 is the string tension. As a result, in the correction obtained the limit of small quark mass ($M_0/m \rightarrow \infty$) is equivalent to the large R limit ($M_0 R \rightarrow \infty$), and the limit of heavy quarks ($M_0/m \rightarrow 0$) is the same as the small R limit. At large R (or at small m) the correction to the quark potential reduces to the constant determined by m and M_0^2 plus terms of order as R^{-2} and higher. Thus, in the framework of the perturbative calculation, the universal Lüscher $1/R$ -term is preserved in the Nambu–Goto string with massive ends too. At small R (or in the case of heavy quarks at string ends) the mass quark correction to the potential, as one would expect, vanishes. In our calculation, this is a direct consequence of the renormalization procedure. It is important that the subtraction procedure used for this aim proves to be unique.

Variational calculation of the potential generated by the Nambu–Goto string with massive ends in the limit $(D - 2) \rightarrow \infty$ results in the radical expression (see Eq. (4.10)). Quark mass contribution to it can be interpreted as a substitution of the critical distance $R_c^2 = \pi(D - 2)/12 M_0^2$ from the Nambu–Goto string with fixed ends by quantity $\bar{R}^2(m, R)$ dependent on quark mass m and distance R . As a result the radical formula (4.10) has sense not only at $R > R_c$, as in the case of the Nambu–Goto string with fixed ends, but also at $R_c^{eff} < R < R_c$ where R_c^{eff} is an effective critical distance determined by condition $V(R_c^{eff}) = 0$. Thus, R_c^{eff} turns out to be dependent on quark mass m and when $m \rightarrow 0$, $R_c^{eff}(m)$ decreases.

In the rigid string model with massive ends the interquark potential is calculated in one-loop approximation. When confining to the quadratic approximation in the string action in this model, the dynamical variables (string position vector) can be presented as a sum of two terms, $\mathbf{u}(t, r) = \mathbf{u}_1(t, r) + \mathbf{u}_2(t, r)$, where $\mathbf{u}_1(t, r)$ is a solution to the Nambu–Goto string with massive ends and $\mathbf{u}_2(t, r)$ is additional variable caused by the curvature in the Polyakov–Kleinert action. It is remarkable that quark masses only affect on $\mathbf{u}_1(t, r)$. This essentially simplifies the

problem under consideration and enables us to use directly the results for potential derived in the Nambu–Goto string with massive ends. In one-loop approximation, the variables $\mathbf{u}_1(t, r)$ and $\mathbf{u}_2(t, r)$ give additive contribution to the interquark potential generated by rigid string. It is true both in the case of the fixed string ends and for the rigid string with massive ends. As a result, the mass quark correction to the one-loop potential generated by rigid string is reduced to the modification of the contribution from the variable $\mathbf{u}_1(t, r)$: the one-loop potential in the Nambu–Goto string with massive ends calculated before should be used here.

In all these calculations we are dealing with an infinite sum of all eigenfrequencies of the Nambu–Goto string with massive ends. It substitutes the well know sum over all integer frequencies in the Nambu–Goto string with fixed or free ends: $(1/2) \sum_{n=1}^{\infty} n = -1/24$. For obtaining a finite value of this new sum, a mode-by-mode subtraction procedure is proposed: each initial eigenfrequency is subtracted by the corresponding frequency of the same string with massive ends taking the limit $R \rightarrow 0$. This prescription turns out to be completely unique. By making use of the argument principle theorem, we present this regularized sum in a form of the integral. Numerical calculation of the subtracted sum over string eigenfrequencies directly and using for this purpose the integral representation give the same result.

The layout of the paper is as follows. In Section 2 the quadratic approximation for the Nambu–Goto string model with massive ends is developed. Upon linearization of the equations of motion and boundary conditions, the general solution to them is obtained. The eigenfrequencies of the string oscillations are determined by a transcendental equation. Then the canonical quantization is shortly outlined. In Section 3, the interquark potential generated by the Nambu–Goto string with massive ends is calculated in one-loop approximation of the perturbation theory. In order to remove the divergence, a new subtraction procedure was proposed. In Section 4, interquark potential, generated by the Nambu–Goto string with massive ends, is calculated by making use of a variational estimation of the functional integral in the limit when $(D - 2) \rightarrow \infty$. In Section 5, the rigid string model with massive ends is treated. By making use of a quadratic approximation for the Polyakov–Kleinert action, the linear equations of motion and boundary conditions are derived. Then canonical quantization of this model is developed. And finally, interquark potential generated in this string model is calculated in one-loop approximation. In Conclusion (Section 6) the obtained results are shortly discussed and possible extension of them are proposed. Some mathematical details of calculation are presented in Appendices A and B.

2 Nambu–Goto string with massive ends

The action of the Nambu–Goto string with point-like masses attached to its ends is written in the following way [15]

$$S = -M_0^2 \iint_{\Sigma} d\Sigma - \sum_{a=1}^2 m_a \int_{C_a} ds_a, \quad (2.1)$$

where $d\Sigma$ is infinitesimal area of the string world surface, C_a ($a = 1, 2$) are the world trajectories of the string massive ends and M_0^2 is the string tension with the dimension of the mass squared ($\hbar = c = 1$).

For our calculation it will be convenient to use the Gauss parametrization of the string world surface:

$$\begin{aligned} x^\mu(\xi) &= (t, r; x^1(t, r), \dots, x^{D-2}(t, r)) = \\ &= (\xi^i; \mathbf{u}(\xi^i)), \quad i = 0, 1. \end{aligned} \quad (2.2)$$

The vector field $u^j(t, r)$, $j = 1, \dots, D - 2$ corresponds to $D - 2$ transverse components of x^μ , while the $\xi = (t, r)$ are the coordinates on the string world sheet. The infinitesimal area $d\Sigma$ is given by $d\Sigma = dt dr \sqrt{g}$, where g is the determinant of the induced metric on the world surface of the string, $g_{ij} = \partial_i x^\mu \partial_j x_\mu$. The metric of the D -dimensional space-time has the signature $(+, -, \dots, -)$.

In this parametrization, the induced metric g_{ij} in the quadratic approximation has the following components

$$g_{ij} = \delta_{ij} - \mathbf{u}_i \mathbf{u}_j, \quad (2.3)$$

where $\mathbf{u}\mathbf{u} = \sum_{j=1}^{D-2} u^j u^j$. From Eq. (2.3) we obtain

$$g^{ij} = g^{-1}[(1 - \mathbf{u}_k^2) \delta_{ij} + \mathbf{u}_i \mathbf{u}_j], \quad (2.4)$$

$$g = \det(g_{ij}) \simeq 1 - \mathbf{u}_i^2, \quad (2.5)$$

(here $\mathbf{u}_0 = \partial \mathbf{u} / \partial t = \dot{\mathbf{u}}$, $\mathbf{u}_1 = \partial \mathbf{u} / \partial r = \mathbf{u}'$). The line elements ds_a , $a = 1, 2$, take the form:

$$ds_a \simeq [1 - \frac{1}{2} \dot{\mathbf{u}}^2(t, r_a)] dt. \quad (2.6)$$

After neglecting unimportant constants, the action (2.1) becomes

$$\begin{aligned} S \simeq & -M_0^2 (t_2 - t_1) R + \frac{M_0^2}{2} \int_{t_1}^{t_2} dt \int_0^R dr [\dot{\mathbf{u}}^2(t, r) - \mathbf{u}'^2(t, r)] + \\ & + \sum_{a=1}^2 \frac{m_a}{2} \int_{t_1}^{t_2} dt \dot{\mathbf{u}}^2(t, r_a), \quad r_1 = 0, \quad r_2 = R. \end{aligned} \quad (2.7)$$

The last two terms in Eq. (2.7), important for deriving dynamical equations, can be rewritten as follows

$$\bar{S} \simeq \frac{M_0^2}{2} \int_{t_1}^{t_2} dt \int_0^R dr [\dot{\mathbf{u}}^2(t, r) \varepsilon(r) - \mathbf{u}'^2(t, r)], \quad (2.8)$$

where $\varepsilon(r)$ is the mass density distribution along the string

$$\varepsilon(r) = 1 + \frac{m}{M_0^2} [\delta(r) + \delta(R - r)]. \quad (2.9)$$

For simplicity we assume that the string ends are loaded with equal masses, $m_1 = m_2 = m$.

Equations of motion and boundary conditions can be deduced from (2.7) or (2.8) by relations

$$\frac{\partial}{\partial t} \left(\frac{\partial L_{str}}{\partial \dot{\mathbf{u}}} \right) + \frac{\partial}{\partial r} \left(\frac{\partial L_{str}}{\partial \mathbf{u}'} \right) = 0, \quad (2.10)$$

and

$$\frac{\partial L_{str}}{\partial \mathbf{u}'} - (-1)^a \frac{\partial}{\partial t} \frac{\partial L_a}{\partial \dot{\mathbf{u}}_a} = 0, \quad a = 1, 2, \quad r_a = 0, R. \quad (2.11)$$

L_{str} is the string Lagrangian density and L_a , $a = 1, 2$ are the Lagrangians of the massive ends of the string in Eq. (2.7). Equations of motion are given by

$$\square \mathbf{u} = 0, \quad (2.12)$$

where $\square = \partial^2/\partial t^2 - \partial^2/\partial r^2$, while for boundary conditions we find [16]

$$m\ddot{\mathbf{u}} = M_0^2 \mathbf{u}', \quad r = 0, \quad (2.13)$$

$$m\ddot{\mathbf{u}} = -M_0^2 \mathbf{u}', \quad r = R. \quad (2.14)$$

Equation (2.12) admits the following solutions:

$$u^j(t, r) \sim \alpha^j \exp[i(\omega/R)t] u(r), \quad j = 1, 2, \dots, D-2, \quad (2.15)$$

in which the string length R has been introduced in order that ω be dimensionless. By substituting (2.15) into (2.12), (2.13) and (2.14) one obtains

$$\mathbf{u}''(r) + \frac{\omega^2}{R^2} \mathbf{u}(r) = 0, \quad (2.16)$$

$$\omega^2 \mathbf{u}(0) = -qR \mathbf{u}'(0), \quad (2.17)$$

$$\omega^2 \mathbf{u}(R) = qR \mathbf{u}'(R), \quad (2.18)$$

where q is a dimensionless parameter

$$q = \frac{M_0^2 R}{m}. \quad (2.19)$$

The differential equations (2.16)–(2.18) are linear, therefore each component of the transverse oscillation of the string satisfies the same equations. Hence, we can write the general solution as a superposition of plane wave solutions

$$u^j(t, r) = i \frac{\sqrt{2}}{M_0} \sum_{n \neq 0} \exp[-i(\omega_n/R)t] \frac{\alpha_n^j}{\omega_n} u_n(r), \quad j = 1, \dots, D-2, \quad (2.20)$$

where the amplitudes α_n^j (Fourier coefficients) obey the usual rule of complex conjugation, $\alpha_n = \alpha_{-n}^*$, in order that u_n^j be real. The eigenfunctions $u_n(r)$ in (2.20) are defined by

$$u_n(r) = N_n \left[\cos\left(\omega_n \frac{r}{R}\right) - \frac{\omega_n}{q} \sin\left(\omega_n \frac{r}{R}\right) \right], \quad (2.21)$$

where N_n are normalization constants and the eigenfrequencies ω_n are the roots of the transcendental equation

$$\tan \omega = \frac{2q\omega}{\omega^2 - q^2}. \quad (2.22)$$

On the ω -axis these roots are placed symmetrically around zero. Hence they can be numbered in the following way $\omega_0 = 0$, $\omega_{-n} = -\omega_n$, $n = 1, 2, \dots$. Therefore it will be sufficient to consider only the positive roots. The eigenfunction $u_n(r)$ obey the orthogonality conditions with the weight function $\varepsilon(r)$ [17]

$$\int_0^R dr u_n(r) u_m(r) \varepsilon(r) = R \delta_{nm}, \quad (2.23)$$

while the functions $u'_n(r)$ satisfy the usual orthogonality conditions

$$\int_0^R dr u'_n(r) u'_m(r) = \frac{\omega_n^2}{R} \delta_{nm}, \quad (2.24)$$

where the eigenfrequencies ω_n are solutions of Eq. (2.22).

The density of the canonical momentum $p^j(t, r)$ is defined in standard way

$$p^j(t, r) = \frac{\partial L_{tot}}{\partial \dot{u}^j} = M_0^2 \dot{u}^j(t, r) \varepsilon(r), \quad (2.25)$$

in which L_{tot} is the Lagrangian density in action (2.8). From (2.25) the total momentum of the string is given by

$$P^j(t) = \int_0^R dr p^j(t, r). \quad (2.26)$$

Obviously, in the problem under consideration, we can put $P_j = 0$. The canonical Hamiltonian is defined by

$$\begin{aligned} H &= \int_0^R dr [\mathbf{p}(t, r) \dot{\mathbf{u}}(t, r) - L_{tot}] = \\ &= \frac{M_0^2}{2} \int_0^R dr [\dot{\mathbf{u}}^2(t, r) \varepsilon(r) + \mathbf{u}'^2(t, r)], \end{aligned} \quad (2.27)$$

In terms of the amplitudes α_{nj} it reads

$$H = \frac{1}{R} \sum_{n=1}^{\infty} \sum_{j=1}^{D-2} (\alpha_n^j \alpha_n^{j+} + \alpha_n^{j+} \alpha_n^j). \quad (2.28)$$

When we quantize, $u^j(t, r)$ and its conjugate momentum $p^j(t, r)$ become operators with canonical commutation relations

$$[u^i(t, r), p^j(t, r')] = i \delta^{ij} \delta(r - r'), \quad (2.29)$$

This implies that the Fourier coefficients become operators and satisfy the relations

$$[\alpha_n^i, \alpha_m^j] = \omega_n \delta^{ij} \delta_{n+m,0}, \quad (2.30)$$

$$i, j = 1, \dots, D-2, \quad n, m = \pm 1, \pm 2, \dots$$

The creation and annihilation operators, respectively a_n^+ and a_n , in Fock space are introduced in usual way

$$\alpha_n^j = \sqrt{\omega_n} a_n^j, \quad \alpha_n^{j+} = \sqrt{\omega_n} a_n^{j+}, \quad (2.31)$$

$$[a_n^i, a_m^{j+}] = \delta^{ij} \delta_{nm}, \quad n, m = 1, 2, \dots, \quad (2.32)$$

and through them, the Hamiltonian (2.28) takes the form

$$H = \frac{1}{R} \sum_{n=1}^{\infty} \sum_{j=1}^{D-2} \omega_n a_n^{j+} a_n^j + \frac{D-2}{2} \frac{1}{R} \sum_{n=1}^{\infty} \omega_n. \quad (2.33)$$

The last term in (2.33) is the usual Casimir energy [18, 19].

3 One-loop potential generated by Nambu–Goto string with massive ends

In this section we shall investigate of the interquark potential arising from the action (2.1), via the perturbation calculation. For this purpose the Euclidean version of the model under consideration should be treated.

Up to the second order in \mathbf{u} the string action (2.1) in Euclidean space is given by ($t_1 = 0, t_2 = T$)

$$S_E = M_0^2 \int_0^T dt \int_0^R dr \left[1 + \frac{1}{2} \mathbf{u} \partial^2 \mathbf{u} \right] + \sum_{a=1}^2 \frac{m_a}{2} \int_{t_1}^{t_2} dt \dot{\mathbf{u}}^2(t, r_a), \quad (3.1)$$

$$r_1 = 0, \quad r_2 = R,$$

where $\partial^2 = \partial^2/\partial t^2 + \partial^2/\partial r^2$. It generates one-loop Feynman diagrams in perturbation theory. The potential $V(R)$ between massive quark separated by a distance R is defined in terms of functional integral as follows [20, 21]:

$$e^{-TV(R)} = \int [D\mathbf{u}] e^{-S_E[\mathbf{u}]}, \quad T \rightarrow \infty. \quad (3.2)$$

From action (3.1), after functional integration, the interquark potential reads

$$V(R) = \lim_{T \rightarrow \infty} \frac{1}{T} \left[\int_0^T dt \int_0^R dr M_0^2 + \frac{D-2}{2} \text{Tr} \ln G^{-1} \right]. \quad (3.3)$$

In Eq. (3.3) $G^{-1} = \partial^2$ is the operator generated by the quadratic part of the action S_E and G is the Euclidean Green function

$$\partial^2 G(x, x') = \delta(x, x'). \quad (3.4)$$

In the momentum space this function is written as

$$G(k_0, k_n) = \frac{1}{k_0^2 + k_n^2}, \quad (3.5)$$

where k_n are admissible values of the wave vector for the field $\mathbf{u}(t, r)$, determined by the boundary conditions (2.13) and (2.14)

$$k_n = \frac{\omega_n}{R}. \quad (3.6)$$

As before ω_n are the roots of frequency equation (2.22). By using the definition of the Tr in the momentum space

$$\text{Tr} \dots = T \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \sum_n \dots, \quad \omega = k_0, \quad (3.7)$$

where the sum is spread over all the discrete values of the component k_n , we can present the functional trace in Eq. (3.3) in the following way

$$\text{Tr} \ln G^{-1} = T \sum_n \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \ln(\omega^2 + k_n^2). \quad (3.8)$$

In the analytical regularization the ω -integration can be performed by the formula [6]

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \ln(\omega^2 + a^2) = \sqrt{a^2}. \quad (3.9)$$

It follows that up to the one-loop level the interquark potential (3.3) is given by

$$V(R) = M_0^2 R + \frac{D-2}{R} v(q), \quad (3.10)$$

in which

$$v(q) = \frac{1}{2} \sum_{n=1}^{\infty} \omega_n(q), \quad (3.11)$$

and $\omega_n(q)$ are the roots of Eq. (2.22).

The sum in (3.11) diverges [18, 19] so that it is necessary to develop a subtraction procedure physically acceptable in order to get the renormalized function $v^{ren}(q)$ in a unique way. We propose here a mode-by-mode subtraction procedure. To this end we consider the asymptotic of eigenfrequencies $\omega_n(q)$ for large n but fixed q [22]

$$\omega_n(q) \sim n\pi + \frac{2q}{n\pi} - \left(1 + \frac{q}{6}\right) \frac{(2q)^2}{(n\pi)^3} + O(n^{-5}), \quad n \rightarrow \infty. \quad (3.12)$$

The divergence of the sum (3.11) is due to the first two terms in (3.12). At the same time we consider the asymptotic forms of Eq. (2.22) at $q \rightarrow 0$

$$\tan \bar{\omega} = \frac{2q}{\bar{\omega}}, \quad (3.13)$$

and at $q \rightarrow \infty$

$$\cot \tilde{\omega} = -\frac{q}{2\tilde{\omega}}. \quad (3.14)$$

The asymptotic of the roots of these equations for large n and fixed q are given, respectively, by

$$\bar{\omega}(q) \sim n\pi + \frac{2q}{n\pi} - \left(1 + \frac{2q}{3}\right) \frac{(2q)^2}{(n\pi)^3} + O(n^{-5}), \quad n \rightarrow \infty, \quad (3.15)$$

and

$$\tilde{\omega}(q) \sim n\pi + \frac{q}{2n\pi} + \frac{q^3}{24n^3\pi^3} + O(n^{-5}), \quad n \rightarrow \infty. \quad (3.16)$$

Thus we see that in order to remove the divergencies in sum (3.11) we must substitute ω_n by $\omega_n - \bar{\omega}_n$. The roots of Eq. (3.14) cannot be used for this purpose in view of their asymptotic (3.16). Therefore the subtraction procedure proposed here is unique.

In terms of the function $v(q)$ the proposed renormalization procedure is defined in the following way

$$v^{ren}(q) = v(q) - \overline{\lim}_{q \rightarrow 0} v(q) + v_0. \quad (3.17)$$

The symbol $\overline{\lim}_{q \rightarrow 0} v(q)$ means that we must take the asymptotic expression of the function $v(q)$. Analogous renormalization scheme is used by calculating the Casimir energy in some field models with special boundary conditions. We have introduced here a constant v_0 in order to satisfy the requirement that at $m = \infty$ Eq. (3.17) must give $v^{ren}(q)$ for string with fixed ends, i.e.

$$v^{ren}(q) = v(q) - \overline{\lim}_{q \rightarrow 0} v(q) + v(q)|_{m=\infty}. \quad (3.18)$$

The frequencies of the string with fixed ends are

$$\omega_n = n\pi. \quad (3.19)$$

Therefore the last terms in (3.18) is defined by

$$v_0 = v(q)|_{m=\infty} = \frac{\pi}{2} \sum_{n=1}^{\infty} n. \quad (3.20)$$

This sum can be renormalized by making use of the Riemann zeta-function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$:

$$v(q)|_{m=\infty} = \frac{\pi}{2} \sum_{n=1}^{\infty} n = \frac{\pi}{2} \zeta(-1) = -\frac{\pi}{24}. \quad (3.21)$$

The final expression for the renormalized function $v^{ren}(q)$ is given by

$$v^{ren}(q) = -\frac{\pi}{24} + \frac{1}{2} \sum_{n=1}^{\infty} [\omega_n(q) - \bar{\omega}_n(q)]. \quad (3.22)$$

This function has been calculated numerically. As shown in Fig. 1, $v^{ren}(q)$ is a monotone increasing function of the dimensionless parameter q .

The interquark potential (3.10) in terms of the function $v^{ren}(q)$ reads

$$V(R) = M_0^2 R + \frac{D-2}{R} v^{ren}(q). \quad (3.23)$$

Equation (3.23) shows that the potential generated by string with massive ends depends on the quarks masses through the dimensionless parameter $q = M_0^2 R/m$. Taking into account this fact we can give here a very clear physical meaning of the subtraction procedure proposed above for obtaining renormalized quantum correction to the potential (3.22). The limit $q \rightarrow 0$ may be obtained by putting $R \rightarrow 0$. Therefore, $v^{ren}(q)$ is determined as a difference of $v(q)$ in two points: $r = R$ and $r \rightarrow 0$. In the same way, the potential (3.23) is equal to the work that should be done for removing quarks connected by string to the distance R , provided that at the beginning they were very closed to each other.

Certainly, this renormalization procedure should be followed by substitution for M_0^2 , the renormalized (physical) value of the string tension. For simplicity we do not introduce new notations and will keep this point in mind.

Now we propose a different way for calculating the sum in Eq. (3.22). It utilize an analytical method that employs the following integral formula from the complex analysis [23].

Let us consider an analytical function $f(z)$ with zeroes of order n_k at points $z = a_k$ and with poles of order p_l at points $z = b_l$ in a region bounded by a contour C . From Cauchy's theorem it follows that

$$\frac{1}{2\pi i} \oint_C dz z \frac{f'(z)}{f(z)} = \frac{1}{2\pi i} \oint_C dz z [\ln f(z)]' = \sum_k n_k a_k - \sum_l p_l b_l. \quad (3.24)$$

In order to obtain the renormalized function $v^{ren}(q)$ through Eq. (3.24), one has to choose an appropriate expression for the function $f(z)$. Equation (3.18) implies that $f(z)$ should be taken in the following form

$$f(\omega) = \frac{\omega^2 - q^2 - 2q\omega \cot \omega}{\omega^2 - 2q\omega \cot \omega} \sin \omega. \quad (3.25)$$

Here the numerator is in fact Eq. (2.22) and the denominator is the same equation in the limit $q \rightarrow 0$ (Eq. (3.13)). The multiplier $\sin \omega$ is introduced to take into account the frequencies (3.19).

When applying the formula (3.24) to nominator, denominator and multiplier $\sin \omega$ in (3.25) separately with the counter C encircling the real positive semiaxis in the complex plane ω , the contribution of the poles into (3.24) is obviously absent. Now we deform the contour C so that it transforms into the imaginary axis on the complex plane. After integrating by parts, the final expression for the renormalized function $v^{ren}(q)$ becomes (Appendix A)

$$v^{ren}(q) = -\frac{\pi}{24} + I(q), \quad (3.26)$$

where

$$I(q) = \frac{1}{2\pi} \int_0^\infty dy \ln \left[1 + \frac{1}{(y/q)^2 + 2(y/q) \coth y} \right]. \quad (3.27)$$

The integral $I(q)$ has been calculated numerically for different values of the parameter q . The results obtained for $v^{ren}(q)$ by making use of (3.22) and (3.26), (3.27) are certainly the same, (see Fig. 1).

Finally we analyze the asymptotic expansion of the potential $V(R)$ in the limit $q \rightarrow \infty$ or $R \rightarrow \infty$. In this region the renormalized function $v^{ren}(q)$, Eq. (3.26), is given by (see Appendix B)

$$v^{ren}(q) \simeq -\frac{\pi}{24} + \frac{\ln 2}{\pi} q + \frac{1}{2\pi q}, \quad q = \frac{M_0^2 R}{m}, \quad q \rightarrow \infty. \quad (3.28)$$

At large R the interquark potential (3.23) can be written up to second order in $1/R$ as follows

$$\begin{aligned} V(R) \simeq & M_0^2 R + \frac{(D-2) \ln 2}{\pi} \frac{M_0^2}{m} - \frac{\pi(D-2)}{24} \frac{1}{R} + \\ & + \frac{D-2}{2\pi} \frac{m}{M_0^2} \frac{1}{R^2} + O(R^{-4}), \quad R \rightarrow \infty. \end{aligned} \quad (3.29)$$

The first terms in (3.29) is the usual linearly rising term $M_0^2 R$ (confining potential). The second term is a constant dependent on the quark mass. The third term is the universal Lüscher $1/R$ -term and the last one is a correction to the string potential due to the finite quark mass.

4 Variational estimation of the string potential

In preceding Section the string potential has been calculated by using the perturbative theory for an arbitrary dimension of space-time D . Otherwise this potential can be derived in the limit $D \rightarrow \infty$ by making use of the variational estimation of the functional integral [1], [4]–[9].

Let us turn to the initial equation (3.2) determining the string potential

$$e^{-TV(R)} = \int [D\mathbf{u}] e^{-S_E[\mathbf{u}]}, \quad T \rightarrow \infty, \quad (4.1)$$

where S_E is the Euclidean action

$$S_E = M_0^2 \int_0^T dt \int_0^R dr \sqrt{\det(\delta_{ij} + \partial_i \mathbf{u} \partial_j \mathbf{u})} + \sum_{a=1}^2 m_a \int_0^T dt \sqrt{1 + \dot{\mathbf{u}}^2(t, r_a)}, \quad (4.2)$$

$$r_1 = 0, \quad r_2 = R.$$

The $1/(D-2)$ -expansion is carried out in standard way [1]. Let us introduce the composite field σ_{ij} for $\partial_i \mathbf{u} \partial_j \mathbf{u}$ and constrain $\sigma_{ij} = \partial_i \mathbf{u} \partial_j \mathbf{u}$ through the Lagrange multiplier α^{ij} . By using the

exponential parametrization of the δ -function, with the understanding that the α^{ij} functional integrals run from $-i\infty$ to $+i\infty$, and calculating the \mathbf{u} -functional integral, Eq. (4.1) becomes

$$e^{-TV(R)} = \int [D\alpha] [D\sigma] e^{-\bar{S}_E[\alpha, \sigma]}, \quad T \rightarrow \infty, \quad (4.3)$$

where

$$\begin{aligned} \bar{S}_E = M_0^2 \int_0^T dt \int_0^R dr & \left[\sqrt{\det(\delta_{ij} + \sigma_{ij})} - \frac{1}{2} \alpha^{ij} \sigma_{ij} \right] + \\ & + \frac{D-2}{2} \text{Tr} \ln(-\partial_i \alpha^{ij} \partial_j). \end{aligned} \quad (4.4)$$

The boundary terms in (4.2) should be taken into account under finding the eigenvalues of the differential operator $-\partial_i \alpha^{ij} \partial_j$ in (4.4). The $1/(D-2)$ -expansion is generated by expanding the action (4.4) around its stationary point. We shall assume that at the stationary point α^{ij} and σ_{ij} are diagonal matrices, $\bar{\alpha}$ and $\bar{\sigma}$ respectively, independent on t and r [1]. From Eqs. (3.6), (3.7) and (3.9) we obtain

$$\begin{aligned} \text{Tr} \ln(-\partial_i \alpha^{ij} \partial_j) &= T \sum_n \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \ln(\alpha^{00} \omega^2 + \alpha^{11} k_n^2) = \\ &= \frac{2T}{R} \sqrt{\frac{\alpha^{11}}{\alpha^{00}}} v(q), \end{aligned} \quad (4.5)$$

where $v(q)$ is defined by (3.11). Now, action (4.4) can be rewritten as

$$\begin{aligned} \bar{S}_E = M_0^2 R T & \left[\sqrt{1 + \sigma_0} \sqrt{1 + \sigma_1} - \frac{1}{2} (\alpha^0 \sigma_0 + \alpha^1 \sigma_1) \right] + \\ & + (D-2) \frac{T}{R} \sqrt{\frac{\alpha^1}{\alpha^0}} v(q), \end{aligned} \quad (4.6)$$

with $\alpha^{ii} = \alpha^i$, $\sigma_{ii} = \sigma_i$, $i = 0, 1$. Equating to zero the variations of the action (4.6) with respect to σ_0 , σ_1 , α^0 , α^1 and solving the corresponding set of algebraic equations allow us to find the stationary points

$$\bar{\alpha}^0 = \sqrt{1 - 2\lambda}, \quad \bar{\alpha}^1 = \frac{1}{\sqrt{1 - 2\lambda}}, \quad (4.7)$$

$$\bar{\sigma}_0 = \frac{\lambda}{1 - 2\lambda}, \quad \bar{\sigma}_1 = -\lambda, \quad (4.8)$$

where the dimensionless function $\lambda(q)$ is defined by

$$\lambda(q) = -\frac{D-2}{M_0^2 R^2} v^{ren}(q). \quad (4.9)$$

Here we have used the renormalized function $v(q)$ from Eq. (3.22).

Putting (4.7) and (4.8) into (4.6) yields the string potential to leading order in $1/(D-2)$:

$$V(R) = M_0^2 R \sqrt{1 - \frac{R_c^2}{R^2} \left(1 - \frac{24}{\pi} I(q)\right)}, \quad (4.10)$$

where $R_c^2 = \pi(D-2)/12M_0^2$ and $I(q)$ is defined in Eq. (3.27). The quark mass contribution in this formula can be interpreted as a substitution of R_c^2 from the Nambu-Goto string with fixed ends by function $\bar{R}^2(m, R)$ dependent on quark mass m and distance R :

$$\bar{R}^2(m, R) = R_c^2 \left[1 - \frac{24}{\pi} I\left(\frac{M_0^2 R}{m}\right)\right]. \quad (4.11)$$

This results in an extension of the applicability of Eq. (4.10) to $R < R_c$. More precisely, formula (4.10) has sense not only for $R > R_c$, that takes place for the string with fixed ends, but also in the region

$$R_c^{eff} < R < R_c, \quad (4.12)$$

where R_c^{eff} is defined by equation

$$\bar{R}^2(m, R_c^{eff}) = (R_c^{eff})^2. \quad (4.13)$$

At $R = R_c^{eff}$ the interquark potential (4.10) vanishes. Now critical distance R_c^{eff} is determined by the quark mass m and when $m \rightarrow 0$ $R_c^{eff}(m)$ also decreases. In Fig. 2 the dimensionless potential $V(R)/M_0$ as a function of the dimensionless distance $\rho = M_0 R$ is plotted for different values of the ratio $\mu = M_0/m$.

At the end of this Section we present an asymptotic expression for the interquark potential (4.10) at $R \rightarrow \infty$. The asymptotic expression for function $I(q)$ when $q \rightarrow \infty$, (or $R \rightarrow \infty$), is given by (B.5)

$$I(q) \simeq \frac{\ln 2}{\pi} \frac{M_0^2 R}{m} + \frac{1}{2\pi} \frac{m}{M_0^2 R}, \quad R \rightarrow \infty. \quad (4.14)$$

Taking into account (4.14), we obtain

$$\begin{aligned} V(R) \sim M_0^2 R + \frac{(D-2) \ln 2}{\pi} \frac{M_0^2}{m} - \frac{\pi(D-2)}{24 R} \left[1 + \frac{12(D-2)(\ln 2)^2}{\pi^3} \frac{M_0^2}{m^2}\right] + \\ + \frac{D-2}{2\pi R^2} \frac{m}{M_0^2} \left[1 + \frac{\pi(D-2) \ln 2}{12} \frac{M_0^2}{m^2}\right] + O(R^{-3}), \quad R \rightarrow \infty. \end{aligned} \quad (4.15)$$

The interquark potential calculated via variational approach, at large values of R has some additional terms as compared with the potential (3.29), obtained by perturbative method. In particular, it turns out that the Lüscher $1/R$ -term

$$\frac{\pi(D-2)}{24 R} \quad (4.16)$$

should be now substituted by

$$\frac{\pi(D-2)}{24 R} \left[1 + \frac{12(D-2)(\ln 2)^2}{\pi^3} \frac{M_0^2}{m^2}\right]. \quad (4.17)$$

5 Quark mass corrections to the rigid string potential

In this Section we calculate the mass quark corrections to the one-loop interquark potential in the framework at the rigid string model [5, 24]. As known, this model can be treated as an effective one taking into account the finite thickness of gluonic tube [25]–[27]. The basic aim of this calculation is to show the principal applicability of the proposed method to the rigid string model with massive ends. Variational estimation of the interquark potential in the framework of this model will be published elsewhere.

The Polyakov-Kleinert action for rigid string with massive ends has the form

$$S = -M_0^2 \iint_{\Sigma} d\Sigma \sqrt{-g} \left[1 - \frac{\alpha}{2} r_s^2 \Delta x^\mu \Delta x_\mu \right] - \sum_{a=1}^2 m_a \int_{\tilde{C}_a} ds_a, \quad (5.1)$$

where the new parameters r_s and α are, respectively, the radius of gluonic tube and a dimensionless constant; Δ is the Laplace- Beltrami operator for the induced metric g_{ij}

$$\Delta = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial \xi^i} \left(\sqrt{-g} g^{ij} \frac{\partial}{\partial \xi^j} \right). \quad (5.2)$$

In the Gauss parametrization (2.2), the operator (5.2), up to the second order in \mathbf{u} , can be written as

$$\Delta \simeq \square + O(\mathbf{u}^2), \quad (5.3)$$

Now action (5.1) reads

$$\begin{aligned} S \simeq & -M_0^2 \int_{t_1}^{t_2} dt \int_0^R dr \left[1 - \frac{1}{2} \dot{\mathbf{u}}^2 + \frac{1}{2} \mathbf{u}'^2 + \frac{\alpha}{2} r_s^2 (\square \mathbf{u})^2 \right] - \\ & - \sum_{a=1}^2 \frac{m_a}{2} \int_{t_1}^{t_2} dt \dot{\mathbf{u}}^2(t, r_a), \quad r_1 = 0, \quad r_2 = R. \end{aligned} \quad (5.4)$$

The equations of motion and boundary conditions for the action (5.4) are

$$(1 + \alpha r_s^2 \square) \square \mathbf{u} = 0, \quad (5.5)$$

$$(1 + \alpha r_s^2 \square) \mathbf{u}' = \frac{m}{M_0^2} \ddot{\mathbf{u}}, \quad r = 0, \quad (5.6)$$

$$(1 + \alpha r_s^2 \square) \mathbf{u}' = -\frac{m}{M_0^2} \ddot{\mathbf{u}}, \quad r = R, \quad (5.7)$$

$$\square \mathbf{u} = 0, \quad r = 0, R, \quad (5.8)$$

($m_1 = m_2 = m$). The Lagrangian in the action (5.4) depends on the first and the second derivatives of the string coordinates, therefore the number of obtained boundary conditions is twice compared with the Nambu-Goto string.

The boundary value problem (5.5)–(5.8) reduces to the two independent boundary problems. Indeed equations of motion are given by product of commuting differential operators $(1 + \alpha r_s^2 \square)$ and \square . Hence, the general solution to this equations can be represented as a sum of two terms

$$\mathbf{u}(t, r) = \mathbf{u}_1(t, r) + \mathbf{u}_2(t, r), \quad (5.9)$$

where

$$\square \mathbf{u}_1 = 0, \quad (5.10)$$

$$\mathbf{u}'_1 = -\frac{m}{M_0^2} \ddot{\mathbf{u}}_1, \quad r = 0, \quad (5.11)$$

$$\mathbf{u}'_1 = +\frac{m}{M_0^2} \ddot{\mathbf{u}}_1, \quad r = R, \quad (5.12)$$

and

$$(1 + \alpha r_s^2 \square) \mathbf{u}_2 = 0, \quad (5.13)$$

$$\mathbf{u}_2(t, 0) = \mathbf{u}_2(t, R) = 0. \quad (5.14)$$

In this case, $\mathbf{u}_1(t, r)$ is the solution for the Nambu-Goto string with massive ends that we have analyzed in Section 2. The string rigidity is taken into account by function $\mathbf{u}_2(t, r)$. The general solution to Eq. (5.13) obeying (5.14) can be presented as

$$u_2^j(t, r) = i \frac{\sqrt{2}}{M_0} \sum_{n \neq 0} \exp \left[i \frac{\nu_n}{R} t \right] \frac{\beta_n^j}{\nu_n} v_n(r), \quad j = 1, 2, \dots, D-2. \quad (5.15)$$

The eigenfunctions $v_n(r)$ are given by

$$v_n(r) = -v_{-n} = N'_n \sin n\pi \frac{r}{R}, \quad n = 1, 2, \dots, \quad (5.16)$$

where N'_n are normalization constants. For the natural frequencies, ν_n , in (5.15) we have

$$\nu_n = -\nu_{-n} = \sqrt{(\pi n)^2 + \frac{R^2}{\alpha r_s^2}}, \quad n = 1, 2, \dots. \quad (5.17)$$

The amplitudes β_n^j satisfy the usual relations of complex conjugation $\beta_n^* = \beta_{-n}$, $n = 1, 2, \dots$.

The Hamiltonian formulation of the model under consideration is developed in the following way. According to Ostrogradskii [28, 29] the canonical variables are defined by

$$q_1^j = u^j, \quad q_2^j = \dot{u}^j, \quad (5.18)$$

$$p_1^j = \frac{\partial L_{tot}}{\partial \dot{u}^j} - p_2^j, \quad p_2^j = \frac{\partial L_{tot}}{\partial \ddot{u}^j}, \quad j = 1, 2, \dots, D-2, \quad (5.19)$$

where L_{tot} is the Lagrangian density in action (5.4)

$$L_{tot} = \frac{M_0^2}{2} \left[\varepsilon(r) \dot{\mathbf{u}}^2 - \mathbf{u}'^2 - \alpha r_s^2 (\square \mathbf{u})^2 \right]. \quad (5.20)$$

Putting L_{tot} and (5.9) into (5.18), (5.19) and taking into account Eqs. (5.10) and (5.13) one obtains

$$\mathbf{q}_1 = \mathbf{u}_1 + \mathbf{u}_2, \quad \mathbf{q}_2 = \dot{\mathbf{u}}_1 + \dot{\mathbf{u}}_2, \quad (5.21)$$

$$\mathbf{p}_1 = M_0^2 [\varepsilon(r) + \alpha r_s^2 \square] \dot{\mathbf{u}}, \quad \mathbf{p}_2 = -\alpha r_s^2 M_0^2 \square \mathbf{u} = M_0^2 \mathbf{u}_2. \quad (5.22)$$

The canonical Hamiltonian is defined by

$$H = \int_0^R dr [\mathbf{p}_1 \dot{\mathbf{q}}_1 + \mathbf{p}_2 \dot{\mathbf{q}}_2 - L_{tot}], \quad (5.23)$$

In terms of Fourier amplitudes it becomes

$$H = \frac{1}{2R} \sum_{n=1}^{\infty} \sum_{j=1}^{D-2} (\alpha_{nj} \alpha_{nj}^+ + \alpha_{nj}^+ \alpha_{nj}) - \frac{1}{2R} \sum_{n=1}^{\infty} \sum_{j=1}^{D-2} (\beta_{nj} \beta_{nj}^+ + \beta_{nj}^+ \beta_{nj}). \quad (5.24)$$

The quantum theory is based on the canonical commutation relations

$$[u_a^i(t, r), p_b^j(t, r')] = i \delta_{ab} \delta^{ij} \delta(r - r'), \quad a = 1, 2, \quad i, j = 1, 2, \dots, D-2, \quad (5.25)$$

or in terms of the Fourier amplitudes

$$[\alpha_n^i, \alpha_m^j] = \omega_n \delta^{ij} \delta_{n+m, 0}, \quad [\beta_n^i, \beta_m^j] = \nu_n \delta^{ij} \delta_{n+m, 0}, \quad n, m = \pm 1, \pm 2, \dots \quad (5.26)$$

By introducing in standard way the annihilation and creation operators

$$\begin{aligned} a_n^i &= (\omega_n)^{-1/2} \alpha_n^i, & a_n^{i+} &= (\omega_n)^{-1/2} \alpha_n^{i+}, \\ b_n^j &= (\nu_n)^{-1/2} \beta_n^j, & b_n^{j+} &= (\nu_n)^{-1/2} \beta_n^{j+}, \\ n &= 1, 2, \dots, & i, j &= 1, 2, \dots, D-2, \end{aligned} \quad (5.27)$$

the Hamiltonian operator (5.24) acquires the following form

$$\begin{aligned} H &= \frac{1}{R} \sum_{n=1}^{\infty} \omega_n \sum_{j=1}^{D-2} a_n^{j+} a_n^j - \frac{1}{R} \sum_{n=1}^{\infty} \nu_n \sum_{j=1}^{D-2} b_n^{j+} b_n^j + \\ &+ \frac{D-2}{2R} \left(\sum_{n=1}^{\infty} \omega_n - \sum_{n=1}^{\infty} \nu_n \right). \end{aligned} \quad (5.28)$$

The last two terms in (5.28) define the Casimir energy in the model under consideration [30]. It is important to note that the second oscillation mode with frequencies ν_n , responsible for the string rigidity, gives a negative contribution to the energy as compared with the oscillation of the first mode with frequencies ω_n . This is also true for the Casimir energy (see the last two terms in (5.28)). It is a direct consequence of the classical expression for the total energy in the rigid string model (5.24). We point out that this defect is typical in all the theories with higher derivatives. To remove it, certainly in formal way only, the quantum states with negative norm are used sometimes [32, 33].

Now we calculate the rigid string potential in one-loop approximation. Again we shall treat the Euclidean version of the model under consideration.

The interquark potential is given by Eq. (3.2) with the Euclidean action ($t_1 = 0$, $t_2 = T$)

$$S_E = M_0^2 \int_0^T dt \int_0^R dr \left[1 + \frac{1}{2} \mathbf{u} (1 + \alpha r_s^2 \partial^2) \partial^2 \mathbf{u} \right]. \quad (5.29)$$

As in section 4, the boundary terms in action (5.4) will be taken into account by finding the proper eigenvalues of the corresponding differential operator.

After functional integration, the potential takes the form

$$V(R) = \lim_{T \rightarrow \infty} \frac{1}{T} \left[\int_0^T dt \int_0^R dr M_0^2 + \frac{D-2}{2} \text{Tr} \ln G^{-1} \right], \quad (5.30)$$

where the operator $G^{-1} = (1 + \alpha r_s^2 \partial^2) \partial^2$ corresponds to the inverse of the Euclidean Green function

$$(1 + \alpha r_s^2 \partial^2) \partial^2 G(x, x') = \delta(x, x'). \quad (5.31)$$

In view of (3.7), the functional trace in Eq. (5.30) can be written as

$$\begin{aligned} \text{Tr} \ln G^{-1} &= \text{Tr} \ln \partial^2 + \text{Tr} \ln(1 + \alpha r_s^2 \partial^2) = \\ &= T \sum_n \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \ln(\omega^2 + k_n^{(1)2}) + T \sum_n \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \ln[1 + \alpha r_s^2 (\omega^2 + k_n^{(2)2})]. \end{aligned} \quad (5.32)$$

$k_n^{(1)}$ are the admissible values of the wave vector for the field $\mathbf{u}_1(t, r)$ determined by the boundary conditions (5.11) and (5.12), $k_n^{(1)} = \omega_n/R$, $n = 1, 2, \dots$, and $k_n^{(2)}$ are the admissible values of the wave vector for the field $\mathbf{u}_2(t, r)$ determined by boundary conditions (5.14), $k_n^{(2)} = \nu_n/R$, $n = 1, 2, \dots$. By using the analytical regularization, formula (3.9), Eq. (5.32) can be reduced to

$$\text{Tr} \ln G^{-1} = \frac{2T}{R} v(q) + \frac{T}{R} \pi \sum_{n=1}^{\infty} \sqrt{n^2 + \epsilon^{-1}}, \quad (5.33)$$

where the dimensionless parameter ϵ is defined by

$$\epsilon = \frac{\alpha}{2} \pi^2 \left(\frac{r_s}{R} \right)^2, \quad (5.34)$$

and the function $v(q)$ is given in (3.11). Now the interquark potential (5.30) assumes the form

$$V(R) = M_0^2 R + \frac{D-2}{R} \left[v(q) + \frac{\pi}{2} \sum_{n=1}^{\infty} \sqrt{n^2 + \epsilon^{-1}} \right]. \quad (5.35)$$

In Eq. (5.35) the second mode of oscillation gives a positive contribution to the energy. As was mentioned above, it means that the formalism, applied here, effectively uses for excitation of the second mode quantum states with negative norm.

To remove the divergences in Eq. (5.35) one has to use a subtraction procedure. The function $v(q)$ has been renormalized in Section 3 (Eqs. (3.22) and (3.26)). Now we renormalize only the last sum in (5.35). It can be rewritten in the following way

$$w(\epsilon) = \frac{1}{2} \sum_{n=1}^{\infty} \sqrt{n^2 + \epsilon^{-1}} = \frac{1}{4} \left(\sum_{n=-\infty}^{+\infty} \sqrt{n^2 + \epsilon^{-1}} - \frac{1}{\sqrt{\epsilon}} \right), \quad (5.36)$$

where the last term removes the term with $n = 0$ in the sum. The renormalized function $w^{ren}(\epsilon)$ is obtained from equation (5.36) by subtracting its value for infinite string [6]

$$\begin{aligned} w^{ren}(\epsilon) &= w(\epsilon) - \lim_{R \rightarrow \infty} w(\epsilon) = \\ &= \frac{1}{4} \left(\sum_{n=-\infty}^{+\infty} - \int_{-\infty}^{+\infty} dn \right) \sqrt{n^2 + \epsilon^{-1}} - \frac{1}{4\sqrt{\epsilon}}. \end{aligned} \quad (5.37)$$

Equation (5.37) can be expressed in terms of the function [30]

$$S(x) = \frac{1}{2} \left(\sum_{n=-\infty}^{+\infty} - \int_{-\infty}^{+\infty} dn \right) \sqrt{n^2 + x} = -\frac{\sqrt{x}}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} K_1(2\pi n \sqrt{x}), \quad (5.38)$$

where $K_1(z)$ is the modified Bessel function [34]. This representation for $S(x)$ is convenient for investigating its behaviour at large x , because the modified Bessel function decays exponentially [34]. For small x another representation of the function $S(x)$ was proposed [6]. By using the definition of the function $S(x)$, Eq. (5.37) becomes

$$w^{ren}(\epsilon) = \frac{1}{2} S(\epsilon) - \frac{1}{4\sqrt{\epsilon}}. \quad (5.39)$$

Finally one-loop calculation of the interquark potential gives

$$V(R) = M_0^2 R + \frac{D-2}{R} \left[v^{ren}(q) - \frac{\pi}{4\sqrt{\epsilon}} - \frac{1}{2\sqrt{\epsilon}} \sum_{n=1}^{\infty} \frac{1}{n} K_1 \left(\frac{2\pi n}{\sqrt{\epsilon}} \right) \right], \quad (5.40)$$

which depends on the quark mass and string rigidity through the dimensionless parameters q and ϵ , respectively.

In the asymptotic limit, $R \rightarrow \infty$, the string potential (5.40) assumes the following form

$$\begin{aligned} V(R) &\simeq M_0^2 R + (D-2) \left(\frac{\ln 2}{\pi} \frac{M_0^2}{m} - \frac{\pi(D-2)}{24R} - \frac{1}{2\sqrt{2\alpha} r_s} \right) + \\ &+ \frac{D-2}{2\pi} \frac{m}{M_0^2} \frac{1}{R^2} - (D-2) \sqrt{\frac{2}{\pi\alpha r_s}} \frac{1}{\sqrt{R}} \exp[-2\sqrt{2} R/\sqrt{\alpha} r_s], \quad R \rightarrow \infty. \end{aligned} \quad (5.41)$$

The first term in the right-hand-side of (5.41) is linearly rising potential, $M_0^2 R$; the second term is a constant determined by quark mass m , string tension M_0^2 and coupling constant $\sqrt{\alpha} r_s$ in the rigid string model action (5.1); the third term is the universal Lüscher $1/R$ -term, the fourth term is the mass quark correction, $(D-2)m/2\pi M_0^2 R^2$. The last term in (5.41) vanishes exponentially when $R \rightarrow \infty$. For different values of the parameters M_0, m, α and r_s the total constant term in (5.41) may, in principle, change sign.

6 Conclusion

In this paper we have developed a consistent method for calculating the interquark potential generated by relativistic string with point-like masses (spinless quarks) at its ends. The obtained results indicate that the correction to the potential due to the finite quark masses turns out to be considerable near the critical (or deconfinement) distance. When the quark mass m decreases the contribution of this correction rises. However, the formula obtained cannot be obviously used at very small m . The point is that all our corrections are based on the linearized dynamical equations in the string theory which are, in some sense, equivalent to the nonrelativistic approximation in the initial string action [16, 30]. At small m , when large velocities of the string ends are important¹, nonrelativistic approximation is certainly not applicable.

The extension of the proposed method for investigating the model of the relativistic string with massive ends at finite temperature is of undoubted interest.

7 Acknowledgments

V.V.N. thanks Prof. H. Kleinert and Dr. A.M. Chervyakov for valuable discussion of some topics considered in this paper. G.L. would like to thank Prof. V.V. Nesterenko and the members of the Bogoliubov Laboratory of Theoretical Physics (JINR) for their kind hospitality during his staying in Dubna. He also acknowledge the financial support from JINR and the Salerno University. This work has been partly supported by the Russian Foundation for Fundamental Research through Project No. 93-02-3972.

APPENDIX A: COUNTER INTEGRAL REPRESENTATION FOR THE SUM OF THE STRING EIGENFREQUENCIES

Here we calculate the sum (3.20) by making use of the counter integral (3.24). Putting in (3.24) $f(z) = \sin \omega$ and deforming counter C as was explained in Section 3, we obtain

$$J = \frac{\pi}{2} \sum_{n=1}^{\infty} n = -\frac{1}{2\pi} \int_0^{+\infty} dy y \coth y. \quad (\text{A.1})$$

The integral (A.1) diverges at the upper limit. To regularize it we introduce into the integrand the cutting multiplier $e^{-\epsilon y}$, $\epsilon > 0$. After doing the integral we take the limit $\epsilon \rightarrow 0$

$$J^{ren} = -\frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} J_{\epsilon}, \quad (\text{A.2})$$

¹It is well known [15] that free (massless) ends of the Nambu-Goto string are permanently moving with light velocity.

where

$$J_\epsilon = \int_0^\infty dy y e^{-\epsilon y} \coth y = \Gamma(2) \left[\frac{1}{2} \zeta\left(2, \frac{\epsilon}{2}\right) - \frac{1}{\epsilon^2} \right], \quad (\text{A.3})$$

and $\zeta(z, q)$ is the Hurwitz zeta-function [34]

$$\zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z}. \quad (\text{A.4})$$

Substituting (A.3) into (A.2) we obtain

$$J^{\text{ren}} = -\frac{\pi}{24} + \lim_{\epsilon \rightarrow 0} \left[-\frac{1}{2\pi\epsilon^2} \right]. \quad (\text{A.5})$$

The pole in (A.5) should be dropped as it is usually done in analytical regularization. Hence

$$J^{\text{ren}} = -\frac{\pi}{24}. \quad (\text{A.6})$$

Application of (3.24) to the fraction in (3.25) alone results, after deforming the counter C and integrating by parts, in (3.27) without any divergences.

APPENDIX B: ASYMPTOTIC EXPANSION OF THE INTEGRAL $I(q)$

In this appendix the asymptotic expansion for $q \rightarrow \infty$ of the function $I(q)$ entered in $v^{\text{ren}}(q)$ will be obtained. In terms of the variable $x = q^{-1}y$, Eq. (3.27) reads

$$I(q) = \frac{q}{2\pi} \int_0^\infty dx \ln \left(1 + \frac{1}{x^2 + 2x \coth qx} \right). \quad (\text{B.1})$$

Let us divide the range of integration into two regions, $(0, 1/q] \cup [1/q, \infty)$, with $q \rightarrow \infty$. The integral in (B.1) can be written as

$$\begin{aligned} I(q) &= \frac{q}{2\pi} \int_0^{1/q} dx \ln \left(1 + \frac{1}{x^2 + 2x \coth qx} \right) + \\ &+ \frac{q}{2\pi} \int_{1/q}^\infty dx \ln \left(1 + \frac{1}{x^2 + 2x \coth qx} \right) = \\ &= \frac{q}{2\pi} J_1(q) + \frac{q}{2\pi} J_2(q). \end{aligned} \quad (\text{B.2})$$

For $x \in (0, 1/q]$ we can substitute $\coth qx$ by $(qx)^{-1}$ and the integral $J_1(q)$ in (B.2) gives

$$J_1(q) \simeq \int_0^{1/q} dx \ln \left(1 + \frac{1}{x^2 + 2/q} \right) \simeq \frac{1}{q} \ln \frac{q}{2} + \frac{1}{q^2}. \quad (\text{B.3})$$

When $x \in [1/q, \infty)$ the function $\coth qx$ can be approximate by 1, so that the integral $J_2(q)$ in (B.2) assumes the following form

$$J_2(q) \simeq \int_{1/q}^{\infty} dx \ln \left(1 + \frac{1}{x^2 + 2x} \right) \simeq 2 \ln 2 - \frac{1}{q} \ln \frac{q}{2}. \quad (\text{B.4})$$

Putting Eqs. (B.3) and (B.4) into (B.2) yields

$$I(q) \simeq \frac{\ln 2}{\pi} q + \frac{1}{2\pi q}, \quad q \rightarrow \infty. \quad (\text{B.5})$$

References

- [1] O. Alvarez, Phys. Rev. D **24**, 440, (1981).
- [2] J. F. Arvis, Phys. Lett. **127B**, 106 (1985).
- [3] V. V. Nesterenko, Teor. Mat. Fiz. **71**, 238 (1987).
- [4] R. D. Pisarski and O. Alvarez, Phys. Rev. D **26**, 3735 (1982).
- [5] H. Kleinert, Phys. Lett. **B174**, 335 (1986); Phys. Rev. Lett. **58**, 1915 (1987).
- [6] H. Kleinert, Phys. Rev. D **40**, 473 (1989).
- [7] G. German and H. Kleinert, Phys. Rev. D **40**, 1108 (1989).
- [8] G. German, Mod. Phys. Lett. **A 6**, 1815 (1991).
- [9] J. Polchinski and Z. Yang, Phys. Rev. D **46**, 3667 (1992).
- [10] A. Barchielli, E. Montaldi and G. M. Prosperi, Nucl. Phys. **B296**, 625 (1988).
- [11] W. Kwong J. L. Rosner and C. Quigg, Annu. Rev. Nucl. Part. Sci. **37**, 325 (1987).
- [12] D. B. Lichtenberg, Int. J. Mod. Phys. **A 2**, 1669 (1987).
- [13] W. Buchmüller, *Fundamental Interaction in Low-Energy System*, ed. F. Dalpiaz, G. Fiorentini and G. Torelli (Plenum, New York, 1984), p. 233.
- [14] M. Lüscher, Nucl. Phys. **B180**, 317 (1981).
- [15] B. M. Barbashov and V.V. Nesterenko, *Introduction to the Relativistic String Theory*, (World Scientific, Singapore, 1990).
- [16] V. V. Nesterenko, Z. Phys. C – Particles and Fields **51**, 643 (1991).
- [17] P. M. Morse and H. Feshback, *Methods of Theoretical Physics* (New York, McGraw - Hill Book Company, 1953).

- [18] B. M. Mostepanenko and N. N. Trunov, *The Casimir Effect and Its Applications* (Moscow, Energoatomizdat, 1990) (in Russian).
- [19] G. Plunien, B. Müller and W. Greiner, Phys. Rep. **134**, 87 (1986).
- [20] K. Wilson, Phys. Rev. D **10**, 2445 (1974).
- [21] M. Lüscher, K. Symanzik and P. Weisz, Nucl. Phys. **B173**, 365 (1980).
- [22] F. W. J. Olver, *Introduction to Asymptotics and Special Functions*, (Academic Press, New York and London, 1974).
- [23] M. A. Lavrent'ev and B. V. Shabat, *Methods of the Theory of Functions with Complex Variables*, (Moscow, Nauka, 1987).
- [24] A. M. Polyakov, Nucl. Phys. **B286**, 406 (1986).
- [25] R. Gregory, Phys. Lett. **B206**, 199 (1988).
- [26] K. Maeda and N. Turok, Phys. Lett. **B202**, 376 (1988).
- [27] V.V. Vladimirov, Yad. Fiz. **57**, 913 (1994).
- [28] M. V. Ostrogradsky, Mem. Ac. St. Petersburg, VI Série, Sci. Math., Phys. et Natur., Vol VI, première partie; Sci. Math. et Phys., Vol. IV, 385, 1850. E. T. Whittaker, *Analytical Dynamics*, (Cambridge, Cambridge University Press, 1937).
- [29] V. V. Nesterenko, J. Phys. **A 22**, 1673 (1989).
- [30] V. V. Nesterenko, Z. Phys. C – Particles and Fields **55**, 265 (1992).
- [31] I. Brevik and I. Clausen, Phys. Rev. D **39**, 603 (1983).
- [32] D. A. Eliezer and R. P. Woodard, Nucl. Phys. **B235**, 389 (1989).
- [33] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).
- [34] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, (New York, Academic Press, 1980).

Figure Captions

Fig. 1. Renormalized sum over the natural frequencies of the string with massive ends, $v^{ren}(q)$, calculated by making use of the definition (3.22) and the integral representation (3.26), (3.27). The results are, certainly, the same.

Fig. 2. Dimensionless interquark potential $V(R)/M_0$ calculated through Eq. (4.10) for different values of ratio $\mu = M_0/m$. The Alvarez result [1] for the Nambu–Goto string with fixed ends is obtained at $\mu = 0$.

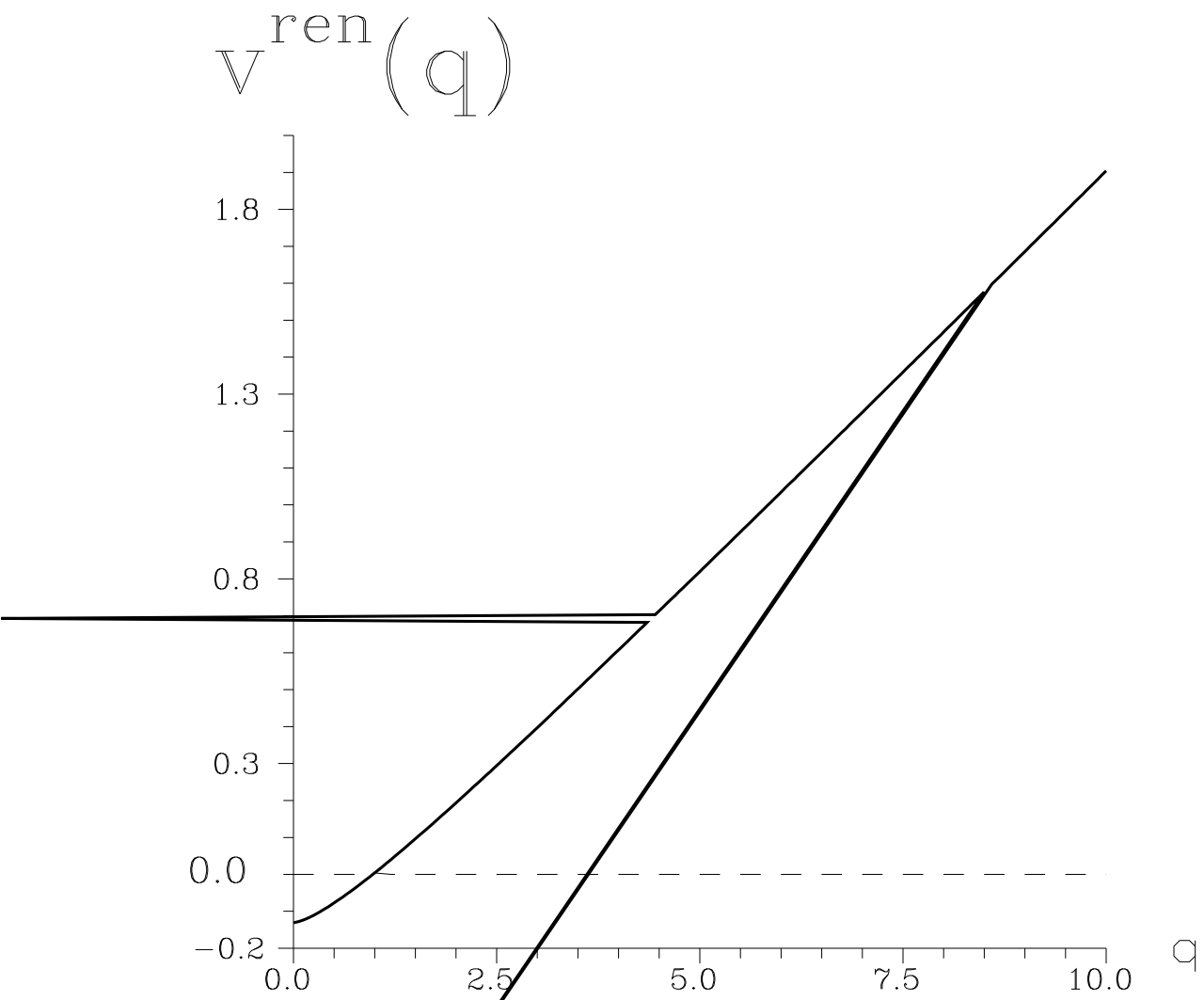


Fig. 1.

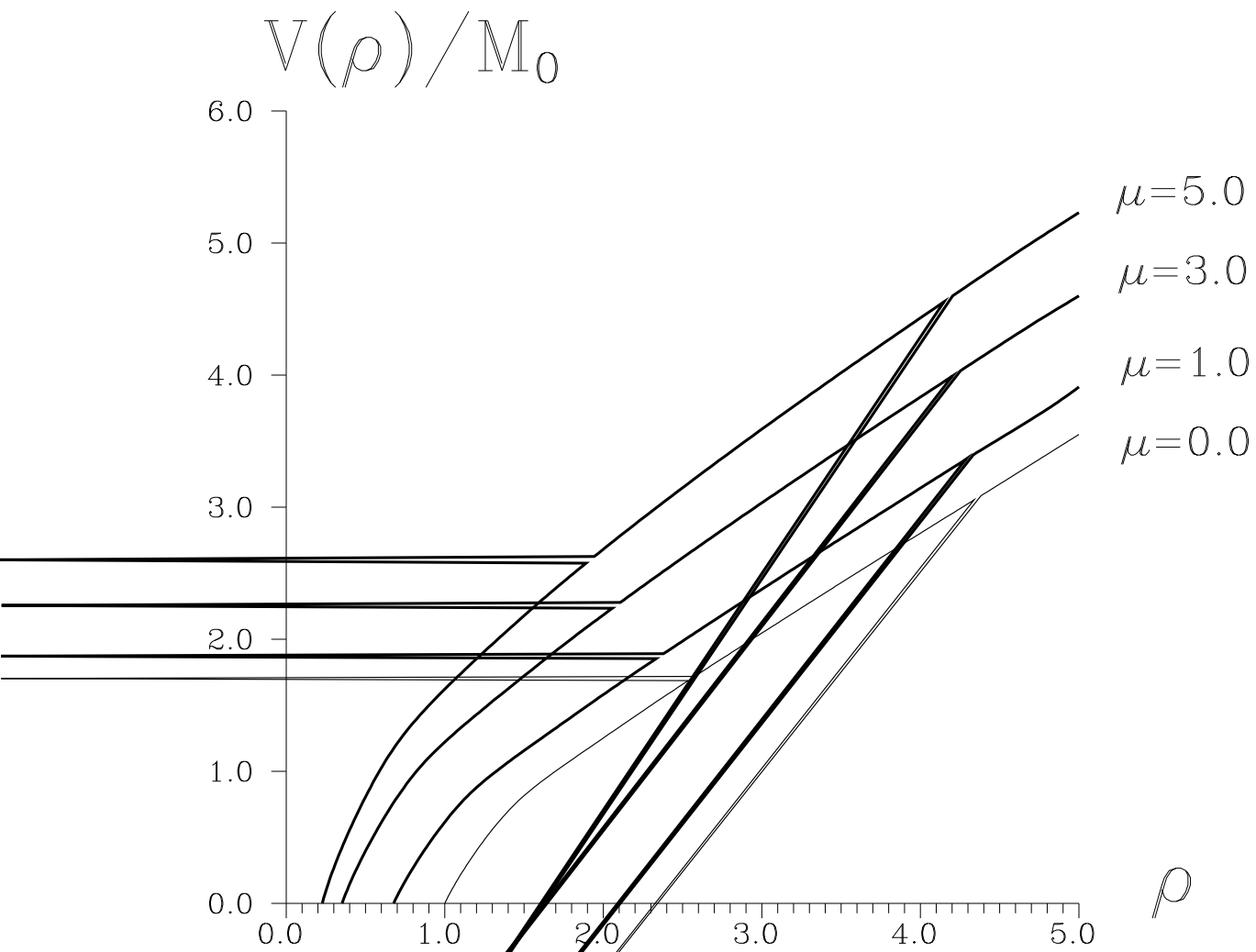


Fig. 2.